Midterm Semester-11/12

Question 1. Solve $y^{'} = xcscy$, $y(0) = \frac{\pi}{2}$ and specify the maximal interval of the type $(-a, a)$ on which your solution is valid.

Solution 1. Solution of the given ODE:

$$-cosy = \frac{x^2}{2} + c,$$

where c is any arbitrary constant. Initial condition $y(0) = \frac{\pi}{2}$ implies $c = 0$.

Therefore the solution is

$$-cosy = \tfrac{x^2}{2}.$$

We know

$$-1 \leq cosy \leq 1.$$

Hence the maximal interval, where the solution is valid is $(-\sqrt{2}, \sqrt{2})$.

Question 2. Solve $y^{'''} - 4y^{''} - 15y^{'} + 18y = 1$ with the initial condition $y(0) = 0, y^{'}(0) = 0$ and $y^{''}(0) = 0$.

Solution 2. The auxiliary equation of the given ODE:

$$\begin{aligned} m^3 - 4m^2 - 15m + 18 &= 0 \\ \text{or} \quad (m-1)(m-6)(m+3) &= 0 \end{aligned}$$

and the roots of this equation are $1, 6$ and -3.

Therefore the complementary function of the ODE is

$$y_{CF} = c_1 e^{x} + c_2 e^{6x} + c_3 e^{-3x}.$$

where c_1, c_2 and c_3 are arbitrary constants. Now using the initial conditions $y(0) = 0, y^{'}(0) = 0$ and $y^{''}(0) = 0$, we have

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 - 3c_2 + 6c_3 &= 0 \\ c_1 + 9c_2 + 36c_3 &= 0 \end{aligned}$$

Clearly, $c_1 = 0, c_2 = 0$ and $c_3 = 0$. Therefore $y_{CF} = 0$.

Let $D = \frac{d}{dx}$. Now the particular integral

$$\begin{aligned} y_{PI} &= \frac{1}{(D-1)(D-6)(D+3)}(1) \\ &= \frac{1}{18}. \end{aligned}$$

Therefore the solution of the ODE is $y = y_{CF} + y_{PI} = \frac{1}{18}$.

Question 2. Solve $(1 + xy - y^3)dx + (x^2 - xy^2 - 2y)dy = 0$ given that there is an integrating factor which is a function of xy.

Solution 3. Write the given ODE as:

$$M(x,y)dx + N(x,y)dy = 0, \tag{1}$$

where $M(x,y) = 1 + xy - y^3$ and $N(x,y) = x^2 - xy^2 - 2y$. Let $g(z)$ be an integrating factor of (1), where $z = xy$. Now the exactness gives

$$\frac{\partial(g(z)M(x,y))}{\partial y} = \frac{\partial(g(z)N(x,y))}{\partial x}.$$

This implies

$$\frac{dg(z)}{dz} = g(z).$$

Therefore $g(z) = e^z$, where $z = xy$. Clearly,

$$d(e^{xy}(x - y^2)) = e^{xy}M(x,y)dx + e^{xy}N(x,y)dy = 0.$$

Hence the general solution is $e^{xy}(x - y^2) = c$, where c is any arbitrary constant.
Question 4. Find a differential equation of the type $a(x)y^{''} + b(x)y^{'} + c(x)y = 0$ which has x and $\frac{1}{x}$ as solutions on $(0, \infty)$, and solve the equation $a(x)y^{''} + b(x)y^{'} + c(x)y = x^3$.
Solution 4. Given $y_1 = x$ and $y_2 = \frac{1}{x}$ are the solutions of the differential equation

$$a(x)y^{''} + b(x)y^{'} + c(x)y = 0 \tag{2}$$

Now Wronksian $W(y_1, y_2; x) = W(x) = -\frac{2}{x}$. Again

$$b(x) = -\frac{W^{'}(x)}{W(x)} = \frac{1}{x}.$$

Since $y_1 = x$ and $y_2 = \frac{1}{x}$ are the solution of (2), we get

$$b(x) + c(x).x = 0$$
$$\implies c(x) = -\frac{1}{x^2}$$

and

$$a(x)\frac{2}{x^3} - \frac{1}{x^3} - \frac{1}{x^3} = 0$$
$$\implies a(x) = 1.$$

Given equation is

$$a(x)y^{''} + b(x)y^{'} + c(x)y = x^3. \tag{3}$$

Putting the values of $a(x), b(x)$ and $c(x)$ in (3), we have

$$x^2y^{''} + xy^{'} - y = x^5,$$

which is a second order Cauchy Euler Equation. Substitute by $x = e^t$, we have

$$(D^2 - 1)y = e^{5t},$$

where $D = \frac{d}{dt}$. Therefore $y_{PI} = \frac{e^{5t}}{24} = \frac{x^5}{24}$.

The general solution is $y = c_1 x + c_2 \frac{1}{x} + \frac{x^5}{24}$.

Question 5. Solve $y^{''} + x^2 y^{'} - xy = 0$ by guessing one solution and then using the Wronskian to find another linearly independent solution.

Solution 5. The given ODE is:

$$y^{''} + x^2 y^{'} - xy = 0. \tag{4}$$

Clearly, $y_1 = x$ is a solution of (4). Let y_2 be another linearly independent solution. Then Wronskian $W(y_1, y_2; x) = W(x) \neq 0$. Again

$$\frac{W^{'}(x)}{W(x)} = -x^2.$$

Therefore $W(x) = ce^{-\frac{x^3}{3}}$. With out loss of generality choose $c = 1$. So

$$W(x) = y_1 y_2^{'} - y_2 y_1^{'} = e^{-\frac{x^3}{3}}.$$

and

$$1 = W(0) = y_1(0)y_2^{'}(0) - y_2(0)y_1^{'}(0) = -y_2(0).$$

Hence we have

$$y_2^{'} - \frac{1}{x} y_2 = \frac{1}{x} e^{-\frac{x^3}{3}}.$$

The integrating factor of the above equation is $\frac{1}{x}$. Therefore

$$y_2 = x\Big(\int \frac{1}{x^2} e^{-\frac{x^3}{3}} dx + c \Big),$$

where c is an arbitrary constant.